University of California, Berkeley

Physics 110A Spring 2003 Section 2 (*Strovink*)

Solution Set 5 (Compiled by Uday Varadarajan)

1. **Griffiths 4.28** The idea here is that the oil will rise to the point at which the gravitational force balances the electrostatic force pulling the dielectric (oil) into the capacitor. The force due to gravity is just the mass of the oil times g,

$$F_g = -\rho\pi(b^2 - a^2)hg. \tag{1}$$

In order to compute the electrostatic force, we need to know the capacitance of a cylindrical capacitor filled with a dielectric. Using Gauss's Law for D in a linear dielctric on a cylindrical surface between the two shells and assuming that a total free charge Q has been placed on the inner conductor,

$$\int \vec{D} \cdot da = \epsilon \int \vec{E} \cdot da = 2\pi\epsilon shE(s) = Q \Rightarrow V - 0 = \int_a^b E(s)ds = \frac{Q}{2\pi\epsilon h}\ln(b/a) \Rightarrow C = \frac{2\pi\epsilon h}{\ln(b/a)}. \tag{2}$$

If l is the height of the coaxial tubes, then we can consider our configuration as a pair of capacitors in parallel, one of height h filled with dielectric and one of height $l - h$ in vacuum, so their total capacitance is just,

$$C_T = \frac{2\pi(\epsilon h + \epsilon_0(l-h))}{\ln(b/a)} = \frac{2\pi(\epsilon_0(1+\chi_e)h + \epsilon_0(l-h))}{\ln(b/a)} = \frac{2\pi\epsilon_0(\chi_e h + l)}{\ln(b/a)}. \tag{3}$$

From Equation 4.64 of Griffiths, the force due to the changing capacitance is just,

$$F_C = \frac{1}{2}V^2\frac{\partial C_T}{\partial h} = \frac{\pi V^2 \epsilon_0 \chi_e}{\ln(b/a)}. \tag{4}$$

Thus, these forces balance if,

$$\rho\pi(b^2 - a^2)hg = \frac{\pi V^2\epsilon_0\chi_e}{\ln(b/a)} \Rightarrow h = \frac{\epsilon_0\chi_e V^2}{\rho g(b^2 - a^2)\ln(b/a)}. \tag{5}$$

2. The basic idea here is that all these particles are just moving in a constant magnetic field, and therefore move in helical orbits which are circular when projected to the xy plane. Further, the frequency of their orbits is the cyclotron frequency $\omega = \frac{eB}{m}$ which is independent of the initial velocities. Thus, they complete their circles at the same time, and therefore refocus at the end of every period. Let's do this quantitatively. The Lorentz force law tells us that,

$$\vec{F} = m\dot{v}_x\hat{\mathbf{x}} + m\dot{v}_y\hat{+}m\dot{v}_z\hat{\mathbf{z}} = e\vec{v} \times \vec{B} = ev_yB_0\hat{\mathbf{x}} - ev_xB_0\hat{\mathbf{y}}. \tag{6}$$

As we only have a magnetic field in the z direction, we see there is no force in the z direction, so $z = v_{z0}t$ for all the particles. In components (with $\omega = \frac{eB}{m}$),

$$\dot{v}_x = \omega v_y, \dot{v}_y = -\omega v_x \Rightarrow \ddot{v}_x = -\omega^2 v_x, \ddot{v}_y = -\omega^2 v_y. \tag{7}$$

The solution to these equations with arbitrary initial x and y velocities is,

$$v_x(t) = v_{0x}\cos(\omega t) + v_{0y}\sin(\omega t) \tag{8}$$
$$v_y(t) = v_{0y}\cos(\omega t) - v_{0x}\sin(\omega t). \tag{9}$$

Integrating this result and using the fact that $x(0) = y(0) = 0$, we see that,

$$x(t) = \frac{1}{\omega}\left(v_{0x}\sin(\omega t) - v_{0y}(1 - \cos(\omega t))\right) \tag{10}$$
$$y(t) = \frac{1}{\omega}\left(v_{0y}\sin(\omega t) + v_{0x}(1 - \cos(\omega t))\right) \tag{11}$$
$$z(t) = v_{0z}t. \tag{12}$$

Thus, we see that regardless of the sign of the charge and the magnitude of the x and y velocities, for $t = \frac{2\pi n}{\omega}$, all the particles focus to the points $(0, 0, \frac{2\pi n v_{0z}}{\omega})$.

3. (a) Using the non-relativistic Galilei transformation law, it is easy to see that for $\vec{\beta} = \frac{E}{cB}\hat{\mathbf{y}}$, we have,

$$\vec{E}' = E\hat{\mathbf{z}} + \frac{E}{cB}cB\hat{\mathbf{y}} \times \hat{\mathbf{x}} = E\hat{\mathbf{z}} - E\hat{\mathbf{z}} = 0. \tag{13}$$

Now, in order to use the non-relativistic transformation law, we must have that the velocity of the new frame is much less than the speed of light, $\beta c = \frac{E}{cB}c \ll c$, which is the approximation we are making.

(b) If a particle is at rest in the $\mathcal{S}$ frame, then it is moving with a velocity, $\vec{v}' = \vec{v} - \vec{\beta}c = -\vec{\beta}c = -\frac{E}{B}\hat{\mathbf{y}}$ in the $\mathcal{S}'$ frame.

(c) Since $E \ll cB$ sp $\beta \ll 1$, so $c\vec{B} \gg \vec{\beta} \times \vec{E}$ and $c\vec{B}' \approx c\vec{B}$. Thus, in the $\mathcal{S}'$ frame, the particle starts at the origin with a velocity $\vec{v}' = -\frac{E}{B}\hat{\mathbf{y}}$ and moves in just a magnetic field $\vec{B} = B_x\hat{\mathbf{x}}$. The solution is just circular cyclotron motion, with a cyclotron radius given by, $R = m\frac{E}{eB^2} = \frac{E}{\omega B}$. In particular, we can just adapt equation (10) to this situation to get,

$$z'(t) = R(1 - \cos(\omega t)) \tag{14}$$
$$y'(t) = R\sin(\omega t). \tag{15}$$

(d) Now, applying a Galilei transformation,

$$\vec{r} = \vec{r}' + \vec{\beta}t \Rightarrow y(t) = y'(t) + \frac{E}{B}t = y'(t) + R\omega t, z'(t) = z(t) \tag{16}$$

we find that,

$$z(t) = R(1 - \cos(\omega t)) \tag{17}$$
$$y(t) = R(\omega t + \sin(\omega t)), \tag{18}$$

which is precisely the cycloid solution found by hard work in Griffiths.

4. We can use, once more, equations (10) and (8) adapted to a magentic field in the y direction with initial velocity p_0/m in the z direction (so $xyz \to zxy$).

$$x(t) = \frac{p_0}{m\omega}(1 - \cos(\omega t)) \tag{19}$$
$$z(t) = \frac{p_0}{m\omega}\sin(\omega t) \tag{20}$$
$$\tag{21}$$

and the corresponding velocities,

$$v_z(t) = (p_0/m)\cos(\omega t) \tag{22}$$
$$v_x(t) = -(p_0/m)\sin(\omega t). \tag{23}$$

Now, note that,

$$v_x(t) = -\omega z(t), \tag{24}$$

so we find that when a particle leaves the bending magnet, so $z(t) = D$, we have that $v_x(t) = -\omega z(t) = -\omega D$, so we get $p_T = |mv_x(t)| = m\omega D = |e|BD$ is independent of the initial momentum. So, for $B_0 = 1T$, $D = 1m$, we have,

$$p_T = 1T \times 1m \times ((3\times10^8 m/s)/c) \times e = (3\times10^8 T\cdot m^2/s)\times e/c = 3\times10^8 V \times e/c = 3\times10^8 eV/c = 0.3GeV/c \tag{25}$$

5. As the divergence of a curl vanishes and by Gauss's law,

$$\nabla \cdot (\nabla \times \vec{B}) = 0 \Rightarrow \nabla \cdot \vec{J} + \frac{\partial}{\partial t}\epsilon_0 \nabla \cdot \vec{E} = \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \tag{26}$$

This is precisely equation 5.29 of Griffiths.

6. We use equation 5.38 in Griffiths and superposition to find the magnetic field,

$$B(z) = \frac{\mu_0 I b^2}{2}\left(\frac{1}{(b^2+(z-a)^2)^{3/2}} + \frac{1}{(b^2+(z+a)^2)^{3/2}}\right) \tag{27}$$

(a) First note that since the first term and the second term are identical except for $a \to -a$, all z derivatives of the second term are identical to those of the first term upon making the substitution $a \to -a$. Second, these derivatives are just sums of terms which are of the form of some polynomial in $z \pm a$ divided by a power of $b^2 + (z \pm a)^2$. Each subsequent derivative either increases or decreases the order of terms of the polynomial by 1. So, for an odd number of derivatives, these polynomials only contain odd powers of $z \pm a$. When we set $z = 0$, this immediately implies that these polynomials contain only odd powers of $\pm a$, and are further identical and therefore they cancel between the two terms.

(b) Now, for an even number of derivatives, these are all even powers of a so the two terms give identical contributions that add. Thus, we only need to compute,

$$\frac{d^2}{dz^2}\left(\frac{1}{(b^2+(z+a)^2)^{3/2}}\right)\bigg|_{z=0} = \frac{d}{dz}\left(\frac{-3(z+a)}{(b^2+(z+a)^2)^{5/2}}\right)\bigg|_{z=0} = \left(\frac{-3(b^2+a^2)+15a^2}{(b^2+a^2)^{7/2}}\right). \tag{28}$$

This vanishes if $b = 2a$.

(c) So now, fix $b = 2a$ and note that the only non-trivial term (after the constant term) in the Taylor expansion comes from the fourth order term. We compute this using Mathematica,

$$\frac{d^4}{dz^4}\left(\frac{1}{(b^2+(z+a)^2)^{3/2}}\right)\bigg|_{z=0} = -\frac{216}{5^{9/2}a^7} \tag{29}$$

Taylor's theorem tells us that,

$$B(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}\frac{d^n B(z)}{dz^n}\bigg|_{z=0}, \tag{30}$$

which to fourth order gives us,

$$B(z) = \frac{4\mu_0 I}{5^{3/2}a}\left(1 - \frac{9z^4}{125a^4}\right) \tag{31}$$

In order to have the magnetic field constant to within 0.1% within some $\Delta z = 0.01m$, we just need

$$\frac{9(0.01m)^4}{125a^4} < 0.001 \Rightarrow a > 0.03m \Rightarrow b > 0.06m. \tag{32}$$

Given this value of $b = 2a$, to get $B = 1T$, we need,

$$B = 0.01T = \frac{4\mu_0 I}{5^{3/2}(0.03m)} \Rightarrow I = 650A. \tag{33}$$

7. (a) For either of these loops, we consider an Amperian loop which goes from $-R$ to R along the z-axis and then returns in a semicircle S_R of radius R in the zy plane. Clearly, the current enclosed is I, and Ampere's law tells us that,

$$\int_{-R}^{R} B_z(z)dz + \int_{S_R} \vec{B}\cdot d\vec{l} = \mu_0 I. \tag{34}$$

Note that along S, $\vec{B} \approx \frac{1}{R^3}$, so $\int_{S_R} \vec{B}\cdot d\vec{l} \approx \frac{1}{R^2}$. Clearly, in the limit where $R \to \infty$, this contribution vanishes and we have,

$$\int_{-\infty}^{\infty} B_z(z)dz = \mu_0 I. \tag{35}$$

(b) When $z \gg b$, we can approximate the fields due to both these loops as magnetic dipoles, and the loop with the larger magnetic dipole moment will produce the larger magnetic field. But the magnetic dipole moment is just proportional to the area, and the area of a square of size $2b$ is $4b^2$ while that of a circle is πb^2, so the square produces the larger field.

(c) For a circle, the magnetic field at the origin is just (from Griffiths 5.38),

$$B(0) = \frac{\mu_0 I}{2b} = 0.5\frac{\mu_0 I}{b}, \tag{36}$$

while that of the square is (four times that of a line segment with $\theta_2 = -\theta_1 = \pi/4$ using Griffiths 5.35),

$$B(0) = 4\frac{\mu_0 I}{4\pi b}(2/\sqrt{2}) = \frac{\mu_0 I\sqrt{2}}{\pi b} = 0.45\frac{\mu_0 I}{b}. \tag{37}$$

Thus, as we get close to the origin, we expect that eventually, the field due to the circle becomes larger than that due to the square.